Calculus Bee today 5PM in TUC 244
(4:30PM refreshments in TUC 300)

From the 2018 Calculus Bee

(1) Find the sum $\sum_{n=0}^{\infty} \frac{1}{2018^n}$, or determine it diverges.

(2) Find $\lim_{x\to\infty} \frac{2^x + x^{2018}}{2^{x+1} + 3x^{2018}}$.

Today is last day for Test #2 corrections

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(1) Geometric series $\left.\begin{array}{l} a = 1^{st}\text{ term} = 1 \\ r = \text{ratio} = \frac{1}{2018} \\ |r|<1 \checkmark \end{array}\right\}$ Sum $\sum_{n=0}^{\infty} a\cdot r^n = \frac{a}{1-r}$

$$= \frac{1}{1-\frac{1}{2018}} = \frac{2018}{2018-1} = \boxed{\frac{2018}{2017}}.$$

(2) $\frac{2^x + x^{2018}}{2^{x+1} + 3x^{2018}} \rightarrow \frac{2^x}{2^{x+1}} = \boxed{\frac{1}{2}}.$

dominant terms as $x\to\infty$

Example (1) Integrate $F(x,y,z) = x^2\sqrt{5-4z}$ over the set $\underbrace{z = 1-x^2-y^2,\ z\geq 0}_{B}$.

Parametrize $x=u,\ y=v$

$\phi(u,v) = (u, v, 1-u^2-v^2)$ (u,v) inside unit disk

(alternately) $\leftarrow u=\theta \quad v=r$

$\psi(u,v) = (v\cos(u), v\sin(u), 1-v^2)$

$0\leq v\leq 1,\ u\in[0,2\pi]$.

$\phi_u = (1,0,-2u)$

$\phi_v = (0,1,-2v)$

$$\phi_u\times\phi_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix}$$

$$= i(0-(-2u)) - j(-2v-0) + k(1)$$

$$= 2ui + 2vj + k = (2u, 2v, 1)$$

$$|\phi_u \times \phi_v| = \sqrt{4u^2+4v^2+1}$$

$$x^2\sqrt{5-4z}$$

$$\iint_B F(x,y,z)\,dA = \int_{u=-1}^{1}\int_{v=\sqrt{1-u^2}}^{\sqrt{1-u^2}} u^2\sqrt{5-4(1-u^2-v^2)}\cdot\sqrt{4u^2+4v^2+1}\,dv\,du$$

$$\underbrace{5-4+4u^2+4v^2}_{1+4u^2+4v^2}$$

better in polar! $u = r\cos\theta$, $v = r\sin\theta$ $\quad dv\,du = r\,dr\,d\theta$

$0 \le \theta \le 2\pi$, $0 \le r \le 1$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\cos^2\theta\sqrt{1+4r^2}\sqrt{4r^2+1}\cdot r\,dr\,d\theta$$

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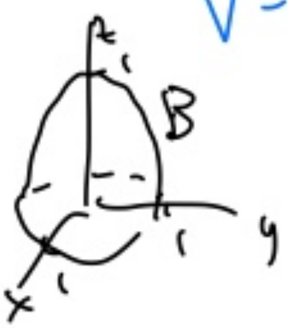
then integrate!

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\cos^2\theta(1+4r^2)\cdot r\,dr\,d\theta$$

$$\cos^2\theta\cdot(r^3+4r^5)\,dr\,d\theta$$

Example ② Calculate the outward (up) flux of $V = (x-y, xz+y, y)$ over the same surface $z = 1-x^2-y^2$, $z \ge 0$.



$$\phi(u,v) = (u, v, 1-u^2-v^2)$$

$$(\phi_u \times \phi_v) = (2u, 2v, 1)$$

from before

$$\iint_B V\cdot\hat{n}\,dA = \iint_{\text{unit disk}} V(\phi(u,v))\cdot(\phi_u\times\phi_v)\,dv\,du$$

$$V = (u-v,\ u(1-u^2-v^2)+v,\ v)$$

$$= \int_{u=-1}^{1} \int_{v=-\sqrt{1-u^2}}^{\sqrt{1-v^2}} (u-v,\ u(1-u^2-v^2)+v,\ v) \cdot (2u,\ 2v,\ 1)\, dv\, du$$

$$(2u^2 - 2uv + 2uv(1-u^2-v^2) + 2v^2 + v$$

$$2u^2 - \cancel{2uv} + \cancel{2uv} - 2u^3v - 2uv^3 + 2v^2 +$$

$$\underset{\text{polar}}{=} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left(2r^2\cos^2\theta - 2r^4\cos^3\theta\sin\theta - 2r^4\cos\theta\sin^3\theta + 2r^2\sin^2\theta + r\sin\theta\right) r\,dr\,d\theta$$

$u = r\cos\theta$ $\quad du\,dv = r\,dr\,d\theta$

$v = r\sin\theta$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left(2r^2 - 2r^4\cos\theta\sin\theta + r\sin\theta\right) r\,dr\,d\theta$$

$$0 \qquad 0$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 2r^3\,dr\,d\theta = \boxed{\pi}$$

$$\left(\frac{1}{2}r^4\Big|_0^1\right)$$

$$\frac{1}{2}$$

Example (2b) Write previous example as differential form integral.

$$V = (x-y,\ xz+y,\ y)$$

$$\iint_B V \cdot \hat{n}\, dA = \iint_B V_1\, dy \wedge dz + V_2\, dz \wedge dx + V_3\, dx \wedge dy$$

$$= \iint_B (x-y)\, dy \wedge dz + (xz+y)\, dz \wedge dx + y\, dx \wedge dy$$

Parametrize again:

$$\begin{cases} x = u \\ y = v \\ z = 1-u^2-v^2 \end{cases} \qquad \begin{aligned} dx &= du \\ dy &= dv \\ dz &= -2u\,du - 2v\,dv \end{aligned}$$

$$= \iint_{\text{unit disk}} (u-v)\, dv \wedge (-2u\,du - 2v\,dv) + \left(u(1-u^2-v^2)+v\right)(-2u\,du - 2v\,dv) \wedge du + v\, du \wedge dv$$

$$= \iint_{\text{unit disk}} (u-v)(2u)\, du \wedge dv + \left(u(1-u^2-v^2)+v\right)(2u)\, du \wedge dv + v\, du \wedge dv$$

$\ldots$

Example ③ Evaluate the outward flux of F through the hemisphere $x^2+y^2+z^2=1$, $z \geq 0$, where $F = \nabla \times A$, where $A = (y+\sqrt{z})\hat{i} + e^{xyz}\hat{j} + \cos(xz)\hat{k}$.

, $\displaystyle\iint_H (\nabla \times A) \cdot \hat{n}\, dA = ?$ surface flux integral.

$$\underset{\text{Stokes Thm}}{=} \oint_{\substack{\partial H \\ \text{ccw}}} A \cdot ds \quad \text{line integral}$$

line integral

$$= \int_0^{2\pi} \left(\sin(t) + 0,\ \underset{1}{e^0},\ \cos(0)\right) \cdot \left(-\sin(t), \cos(t), 0\right) dt$$

$$\alpha(t) = (\underset{x}{\cos(t)}, \underset{y}{\sin(t)}, \underset{z}{0})$$

$$\alpha'(t) = (-\sin(t), \cos(t), 0)$$

$$= \int_0^{2\pi} \left(-\sin^2(t) + \cos(t) + 0\right) dt$$

$$= \int_0^{2\pi} \left(-\frac{1}{2} - \frac{\cos(2t)}{2} + \cos(t)\right) dt$$

$$= -\frac{t}{2} - \frac{\sin(2t)}{4} + \sin(t) \Big|_0^{2\pi}$$

$$= -\frac{2\pi}{2} = \boxed{-\pi}.$$

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surface flux integral

$$\iint_H (\nabla \times A) \cdot \hat{n}\, dA \qquad \nabla \times A = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y+\sqrt{z} & e^{xyz} & \cos(xz) \end{vmatrix}$$

$$= i\left(\partial_y(\cos(xz)) - \partial_z e^{xyz}\right) - j\left(\partial_x(\cos(xz)) - \partial_z(y+\sqrt{z})\right.$$

$$+k\left(\partial_x(e^{xyz}) - \partial_y(y+\sqrt{z})\right)$$

$$= i\left(0 - xye^{xyz}\right) - j\left(-z\sin(xz) - \tfrac{1}{2}z^{-1/2}\right)$$

$$+k\left(yze^{xyz} - 1\right)$$

$$\nabla \times A = \left(-xye^{xyz},\ z\sin(xz) + \frac{1}{2\sqrt{z}},\ yze^{xyz} - 1\right)$$

$$\phi(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right)$$

$$\phi_u = \qquad\qquad \phi_u \times \phi_v = $$

$$\phi_v =$$

We could do it, but....

(3b) Write as differential form integral.

$$\int_H (\nabla \times A)\cdot\hat{n}\, dA = \int_H d\left(A_1\, dx + A_2\, dy + A_3\, dz\right)$$

surface

$$= \int_H \left((A_1)_x dx + (A_1)_y dy + (A_1)_z dz\right)\wedge dx$$

$$+ \cdots$$

$$+ \cdots$$

line integral $\displaystyle\int_{\text{unit circle}} A_1\, dx + A_2\, dy + A_3\, dz$

$$\left.\begin{array}{l} x = \cos\theta \\ y = \sin\theta \\ z = 0 \end{array}\right\} \quad \begin{array}{l} dx = -\sin\theta \, d\theta \\ dy = \cos\theta \, d\theta \end{array}$$

$$A = (y + \sqrt{z})\hat{i} + e^{xyz}\hat{j} + \cos(xz)\hat{k}.$$

$$A_1 = (\sin\theta) \quad A_2 = e^0 = 1, \quad A_3 = 1$$

$$\int_{\theta=0}^{2\pi} (\sin\theta)(-\sin\theta \, d\theta) + (1 \cos\theta) d\theta + (1)(0)$$

$$= \int_0^{2\pi} (-\sin^2\theta + \cos\theta) \, d\theta$$

$$= \int_0^{2\pi} -\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) + \cos\theta \quad d\theta$$

$$= -\frac{1}{2}(2\pi) = \boxed{-\pi}.$$